VOLUME OF INTERSECTION OF SPHERE WITH PENTAGONAL PYRAMID: CASE OF SPECIAL PRACTICAL INTEREST FOR NANO-GRAINED COMPOSITES

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Abstract

Nanocomposite materials are built up from crystalline nanoparticles of different phases. As nanocrystallites often take icosahedral shape the structural model for a bifractional nanocomposite leads to the geometric task: to determine the volume cut out from the sphere by the lateral facets planes of a pentagonal pyramid, top of which coincides with the sphere center. In the paper, this problem is resolved analytically. Based on this result, it is possible to estimate quantitatively the density of bifractional nanocomposites.

1. Introduction

Various problems of science and engineering need the calculation of volume of intersection between spheres and spheres with other figures. In the paper [1], we found the explicit analytical solution of this problem for three spheres in general case, i.e. for different radii of spheres and different distances between their centers. This result is used in computing of materials electron structure. Later we solved [2] the six-sphere problem for special robotics case of practical interest, namely, determined workspace volume of the 6-DOF parallel manipulator "Stewart Platform". Here we intend to consider one more related geometric problem, which is an important issue for the powder metallurgy technologies.

Currently, number of powder materials containing chemical elements such as titanium, aluminum, silicon, boron, carbon, etc. are used as feedstock for producing various important ceramics, cermets, superalloys, etc. both by conventional powder metallurgy and self-propagating high-temperature synthesis (SHS) methods. In such heterogeneous media, heterodiffusion and chemical reactions are initiated at contact points between dissimilar

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particles. Hence, the number of initial heterogeneous contacts significantly affects the rate of these processes.

The paper [**3**] determines the ratio of particle sizes, which is optimal for two-component mixtures to obtain the structures with maximum heterogeneous contacts. According to the model proposed, the starting mixture is considered a random packing of spherical particles of two different sizes. Based on this model, the particle size ratio has been estimated so that the number of heterogeneous contacts between the particles of different phases maximal. In particular, it was analyzed the Ti–B composition, what is important for producing titanium diboride TiB_2. By introducing the IRA (Ideal Reaction Area) concept, reaction mixture – initial charge of two reagents is considered as a mixture of spherical particles of two different diameters, in which all the big particles are surrounded by the small ones. In this approach, the chemical synthesis process can be investigated within a single IRA and the result expanded to the whole system. As is known, any disordered system of particles is characterized by the structural motif of 5th order symmetry. Then each IRA can be assumed to be surrounded by 12 similar IRAs in the configuration of slightly distorted icosahedron.

As icosahedron tops are pentagonal pyramid tops as well, above described model leads to the geometric task to determine the volume cut out from the sphere by the lateral facets planes of the pentagonal pyramid, top of which coincides with the sphere center. In the paper, this problem is resolved analytically.

2. Volume of intersection between sphere and pentagonal pyramid

Let us introduce the necessary designations:R is the sphere radius or, what is the same, the edges of the pyramid; r is the radius of the circle ascribed around the pyramid basal pentagon or, what is the same, the distance from the pentagon center to its vertices; h is the radius of the circle inscribed in this pentagon or, what is the same, the distance from its center to its sides; and $\frac{\pi}{5}$ is the angle between r and h line-segments. From **Figure 1**, it follows the relations:

$$r = \frac{R}{2\sin\frac{\pi}{5}}, \tag{1}$$

$$h = \frac{R}{2\,\mathrm{tg}\frac{\pi}{5}}, \tag{2}$$

and

$$r^2 - h^2 = \frac{R^2}{4}. \tag{3}$$

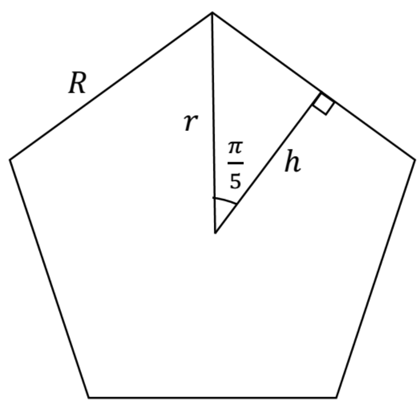

Figure 1. Pyramid basal pentagon.

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Pyramid height H (**Figure 2**), base area S_{Pyramid} , and volume V_{Pyramid} , respectively, are equal to:

$$H = \sqrt{R^2 - r^2}\,, \tag{4}$$

$$S_{\text{Pyramid}} = 10 \cdot \frac{1}{2} \cdot \frac{R}{2} \cdot h = \frac{5Rh}{2}\,, \tag{5}$$

and

$$V_{\text{Pyramid}} = \frac{S_{\text{Pyramid}} H}{3} = \frac{5RHh}{6}\,. \tag{6}$$

Since the height of the spherical segment resting on the pyramid basal plane (**Figure 3**) is equal to $R - H$, the volume of this spherical segment is:

$$V_{\text{Segment}} = \pi(R - H)^2 \left(R - \frac{R-H}{3}\right) = \frac{2\pi R^3}{3} - \pi R^2 H + \frac{\pi H^3}{3}\,. \tag{7}$$

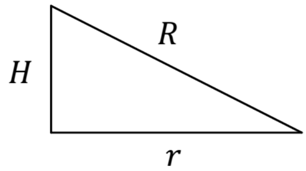

Figure 2. For calculation of pentagonal pyramid height.

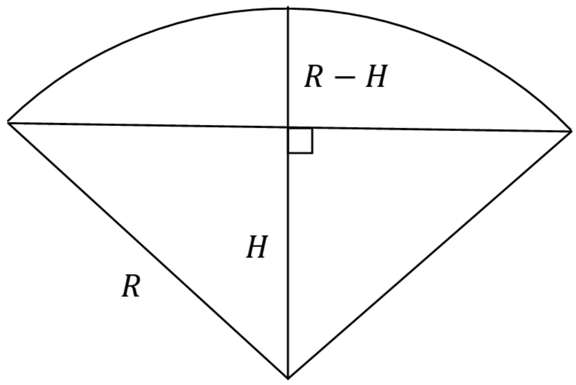

Figure 3. Section of spherical segment resting on pyramid basal plane.

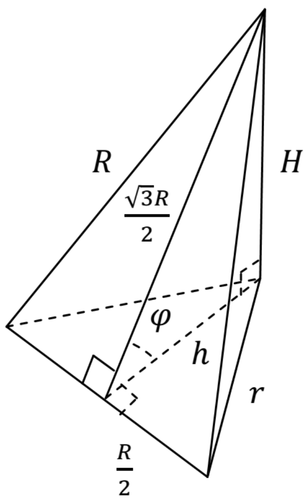

Figure 4. For calculation of angle between pyramid hip facets and basal planes.

Further, since the height of the pyramid triangular hip facets is $\sqrt{R^2 - \left(\frac{R}{2}\right)^2} = \frac{\sqrt{3}R}{2}$ (**Figure 4**), the angle φ between the planes of these facets and pyramid base is determined from the relation:

$$\cos\varphi = \frac{h}{\frac{\sqrt{3}R}{2}} = \frac{1}{\sqrt{3}\,\mathrm{tg}\frac{\pi}{5}}\,. \tag{8}$$

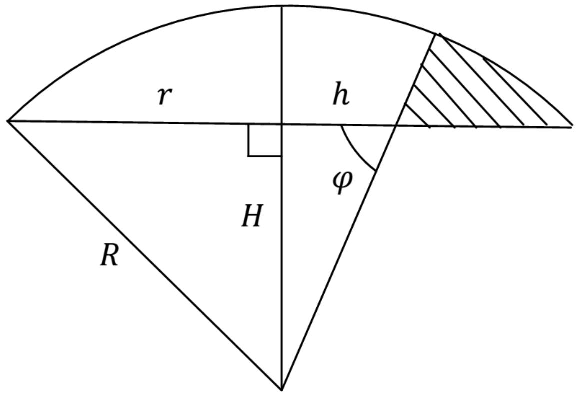

Figure 5. Cross-section of figure in form of "apple slice" in equatorial plane of sphere.

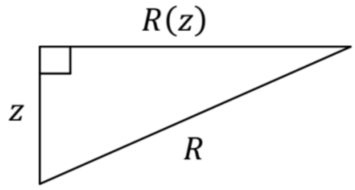

Figure 6. For calculation of $R(z)$ radius.

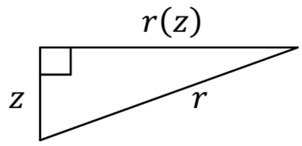

Figure 7. For calculation of $r(z)$ radius.

Let us consider the cross-section (**Figure 5**) of the figure in form of "apple slice" in the equatorial plane of the sphere, in which the corresponding hip facet of the pyramid lies. Let z denotes the coordinate along the "apple slice" edge or, equivalently, along the corresponding side of the pyramid basal pentagon. In **Figures 6** and **7**, the radii $R(z)$ and $r(z)$ at the height z are calculated using following formulas:

$$R(z) = \sqrt{R^2 - z^2} \tag{9}$$

and

$$r(z) = \sqrt{r^2 - z^2}\,. \tag{10}$$

Obviously, $r(z) \geq h$ and, consequently, $z^2 \leq r^2 - h^2 = \left(\frac{R}{2}\right)^2$ and $z \leq \frac{R}{2}$. Therefore, z varies from 0 to $\frac{R}{2}$ and to calculate the "apple slice" volume it is necessary to integrate its cross-sectional area at the height z in the interval $0 \leq z \leq \frac{R}{2}$ and then double the result.

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From **Figure 8**, at height $z$the "apple slice" cross-sectional area is equal to the right triangle area plus the area under the curve of the function describing the circle arc and minus the rectangle area.

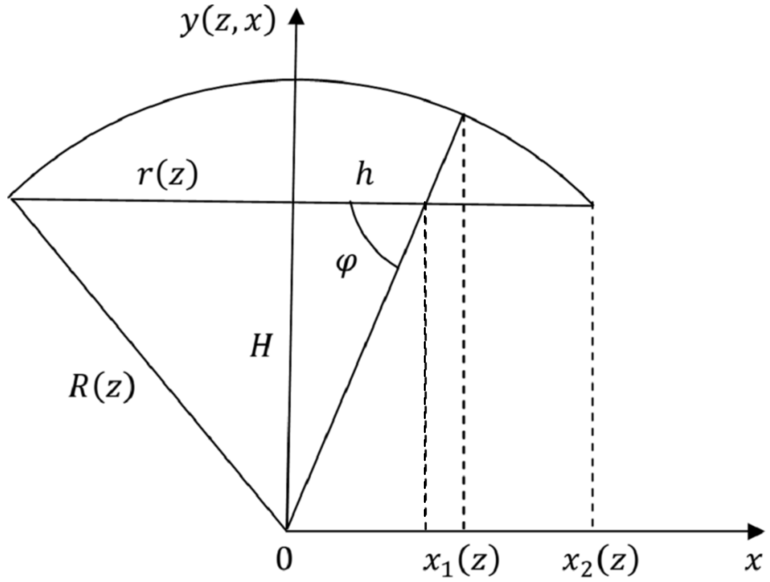

Figure 8. For calculation of cross-section area of figure in form of "apple slice".

Hypotenuse and adjacent to the angle φ leg of the right triangle, respectively, are equal to $R(z)-\sqrt{H^2+h^2}$ and $\left(R(z)-\sqrt{H^2+h^2}\right)\cos\varphi$. Therefore, the area of this right triangle is:

$$S_{\text{Triangle}}(z)=\frac{1}{2}\cdot\left(R(z)-\sqrt{H^2+h^2}\right)\cdot\left(R(z)-\sqrt{H^2+h^2}\right)\cos\varphi\cdot\sin\varphi=$$
$$=\frac{\sin 2\varphi}{4}\left((H^2+R^2)-2\sqrt{(H^2+h^2)(R^2-z^2)}+(R^2-z^2)\right). \quad (11)$$

Rectangle sides are H and $r(z)-\left(h+\left(R(z)-\sqrt{H^2+h^2}\cos\varphi\right)\right)=r(z)-R(z)\cos\varphi$. Therefore, for this rectangle area we obtain the expression:

$$S_{\text{Rectangle}}(z)=H(r(z)-R(z)\cos\varphi)=H\left(\sqrt{r^2-z^2}-\sqrt{R^2-z^2}\cos\varphi\right). \quad (12)$$

Equation of the function of argument x describing the circle arcis:

$$y(z,x)=\sqrt{R^2(z)-x^2}\,. \quad (13)$$

Coordinates of starting and ending points of the arc are determined, respectively, by relations

$$x_1(z)=h+\left(R(z)-\sqrt{H^2+h^2}\right)\cos\varphi \quad (14)$$

and

$$x_2(z)=r(z)\,. \quad (15)$$

Then, using the tables of integrals we obtain the required area under the curve of the function describing the circle arc:

$$S_{\text{Arc}}(z)=\int_{x_1(z)}^{x_2(z)}dx\,y(z,x)=\int_{x_1(z)}^{x_2(z)}dx\,\sqrt{R^2(z)-x^2}=$$
$$=\frac{H}{2}\sqrt{r^2-z^2}-\frac{1}{2}\left(\frac{\pi}{2}-\varphi+\frac{\sin 2\varphi}{2}\right)(R^2-z^2)+\frac{1}{2}(R^2-z^2)\arcsin\sqrt{\frac{r^2-z^2}{R^2-z^2}}\,. \quad (16)$$

As a result, the "apple slice" cross-sectional area at the height z we obtainin form:

$$S(z)=S_{\text{Triangle}}(z)+S_{\text{Arc}}(z)-S_{\text{Rectangle}}(z)=$$
$$=\frac{Hh}{2}-\frac{1}{2}\left(\frac{\pi}{2}-\varphi\right)(R^2-z^2)-H\sqrt{r^2-z^2}+\frac{1}{2}(R^2-z^2)\arcsin\sqrt{\frac{r^2-z^2}{R^2-z^2}}\,. \quad (17)$$

And after a cumbersome integration, in particular, using tables of integrals [**4**] the volume of the figure in form of "apple slice" is calculated as:

$$V_{\text{“Apple Slice”}} = 2\int_0^{\frac{R}{2}} dz\, S(z) = \frac{RHh}{3} - \frac{\pi H r^2}{15} - \frac{2\pi R^2 H}{15} + \frac{2R^3}{3}\,\mathrm{arctg}\frac{3-\sqrt{5}}{2}\,. \quad (18)$$

Hence, the required volume is

$$V = V_{\text{Pyramid}} + V_{\text{Segment}} - 5\,V_{\text{“Apple Slice”}} = \left(1 - \frac{5}{\pi}\,\mathrm{arctg}\frac{3-\sqrt{5}}{2}\right)\frac{2\pi R^3}{3}\,. \quad (19)$$

3. Concluding remarks

As expected, the hemisphere volume,

$$V_{\text{Hemisphere}} = \frac{2\pi R^3}{3}\,, \quad (20)$$

exceeds the volume cut out from the sphere by the lateral facets planes of the pentagonal pyramid, top of which coincides with the sphere center:

$$V \approx 0.419\, V_{\text{Hemisphere}} \approx 0.878\, R^3\,. \quad (21)$$

Based on the obtained expression, it is possible to estimate quantitatively the density of bifractional nanocomposites.

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